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Degeneracy of the Lowest Landau Level and Quantum Group $U_q(sl(2))$ on the Sphere

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We show that the degree of degeneracy of a one-electron system on the surface of a sphere, in the lowest Landau level, can be obtained in a consistent way using the cyclic irreducible representation of the quantum group $U_q\ (sl(2))$. The allowed degree of degeneracy arises naturally from the properties of this representation when q is a root of unity.

A challenge for theoretical condensed matter physicists resides in the explanation of the fractional quantum Hall effect (FQHE) [1], namely, the reason for the appearance of fractional filling factors. The Hall conductivity of a two-dimensional electron system subjected to a strong, uniform magnetic field perpendicular to its plane appears to be quantized [2],

$$\sigma_{\mathrm{H}} = \nu \frac{e^2}{h} \tag{1}$$

where ν is the filling factor, i.e., the ratio between the number of electrons and the degeneracy of the corresponding Landau level. Fractions with an odd denominator have been shown to stem from Coulomb interaction [3]. Recently, however, half-integer filling factors have been found [4]. Although the ground-state wavefunction was admirably guessed by Laughlin [5] many years ago, its origin remains to be explained.

One of the most fruitful theoretical approaches towards understanding the FQHE comes from representation theory, this time from quantum groups [6, 7]. It has been shown that the symmetries exhibited by this phenomenon correspond to the quantum group $U_q(sl(2))$ [8–12]. The parameter q that

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characterizes the deformation of the classical algebra $sl(2)$ is related to the filling factor through

$$q = e^{2\pi i\nu} \tag{2}$$

It has been shown [13, 14] that the degeneracy of the Landau levels can be readily obtained whenever q is a root of unity.

In this paper, using the technique of the quantum group $U_q(sl(2))$, we show that the one-electron system moving on the surface of a sphere in the presence of a monopole magnetic exhibits a degeneracy. Starting from the cyclic irreducible representation at root of unity of the $U_q(sl(2))$, we determine the degree of degeneracy of the lowest Landau level of the system under consideration.

Let us consider a spinless nonrelativistic electron of mass m_e constrained to move on the surface of a sphere of radius R (for simplicity, we take $\hbar = c = 1$ and $R = 1/2$) having a magnetic monopole in its center. The Hamiltonian of the system can be written as [15]

$$H = \frac{2}{m_e}[\mathbf{r} \times (\mathbf{P} + e\mathbf{A})]^2 \tag{3}$$

where $\mathbf{P} = -i\nabla$ and $\nabla \times \mathbf{A} = B\Omega$, $\Omega = \mathbf{r}/R$. In complex notation the above Hamiltonian takes the form

$$H = \frac{2}{m_e}(1 + z\bar{z})^2\,(p_z + eA_z)(P_{\bar{z}} + eA_{\bar{z}}) \tag{4}$$

Introducing

$$p_z = -i\frac{\partial}{\partial_z}, \qquad p_{\bar{z}} = -i\frac{\partial}{\partial_{\bar{z}}}, \qquad eA_z = i\frac{\phi}{2}\frac{\bar{z}}{1 + z\bar{z}} \tag{5}$$

we find that Eq. (4) becomes

$$H = \frac{2}{m_e}(1 + z\bar{z})^2\left(\frac{\partial}{\partial_z} - \frac{\phi}{2}\frac{\bar{z}}{1 + z\bar{z}}\right)\left(\frac{\partial}{\partial_{\bar{z}}} + \frac{\phi}{2}\frac{\bar{z}}{1 + z\bar{z}}\right) - \frac{\phi}{m_e} \tag{6}$$

Before we study the degeneracy, let us give a brief review on the realization of the quantum group $U_q(sl(2))$. We recall that this group is generated by the generators E^+, E^-, K, and K^{-1} satisfying the commutation relations

$$[E^+, E^-] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \qquad KE^{\pm}K^{-1} = q^{\pm 1}E^{\pm} \tag{7}$$

In ref. 16 it is shown that there is a way leading to the realization of the quantum group $U_q(sl(2))$. Starting from the angular momentum operator

$$J = z\frac{\partial}{\partial_z} - \bar{z}\frac{\partial}{\partial_{\bar{z}}} \tag{8}$$

it is possible to construct the generators E^+, E^-, K, and K^{-1}. These generators can be defined as

$$E^+ = -z[J+1]_q, \qquad E^- = z^{-1}[J]_q$$
$$K = q^{J+1/2}, \qquad K^{-1} = q^{-J-1/2} \tag{9}$$

where the quantum symbol $[x]_q$ is

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{10}$$

They satisfy the $U_q(sl(2))$ commutation relations (7) provided that the q-deformed parameter is chosen to be

$$q = e^{2\pi i/(\phi+1)} \tag{11}$$

where $\phi = 4\pi R^2 B$ is the total magnetic flux through the surface of the system.

After this short review of the realization of the quantum group $U_q(sl(2))$ by the help of the angular momentum operator, let us return to the degeneracy of the Landau level of the present system.

We begin by noting that in ref. 16 it is shown that

$$[E^{\pm}, H] \cong 0, \qquad [K^{\pm 1}, H] \cong 0 \tag{12}$$

The commutativity between the $U_q(sl(2))$ and the Hamiltonian is satisfied only on the ground states, hence the symbol $\cong$. Starting from the last result, we consider in the further analysis just the lowest Landau level.

On the other hand, according to quantum mechanics, Eqs. (12) mean in turn that the Hamiltonian of the system is invariant under the quantum group $U_q(sl(2))$, namely the symmetry of the system is $U_q(sl(2))$. Consequently, the ground states of this system are degenerated.

To determine the degree of degeneracy of the ground states of the system under consideration, let us impose a condition on the q-deformed parameter. For this, we require that q to be a kth root of unity, namely

$$q = e^{2\pi i/k} \tag{13}$$

where k is a positive integer value. Comparing Eqs. (11) and (13), we get

$$k = \phi + 1 \tag{14}$$

This choice is related to the fact that the representation of the quantum group when the deformation parameter is a root of unity has many important properties. For instance, it is a k-dimensional and cyclic irreducible representation. In the follows, we exploit the last properties to show the relation between the degree of degeneracy and the representation, through its dimension.

To clarify the connection between the deformation and the degree of degeneracy, according to Eqs. (12) we choose a common basis of the operators $K^{\pm 1}$ and H. This basis can be taken as a set of vectors $|m\rangle$, where m denotes the total number of available states in the lowest Landau level or the ground states. For this, we can take

$$H|m\rangle = E_0|m\rangle, \qquad K^{\pm 1}|m\rangle = q^{\pm(\lambda_1-\lambda_2-m)}|m\rangle \tag{15}$$

where E_0 is the energy of the lowest Landau level.

From the representation theory of the quantum group at a root of unity [17, 18], it is well known that the action of the generators defined in Eqs. (9) on the ground states of the system $|m\rangle$ takes the following form:

$$\begin{aligned}
E^{+}|m\rangle &= (\lambda_1 - \lambda_2 - m + 1)|m-1\rangle, \qquad 1 \leq m \leq k-1 \\
E^{+}|0\rangle &= \lambda_3^{-1}(\lambda_1 - \lambda_2 + 1)|k-1\rangle \\
E^{-}|m\rangle &= |m+1\rangle, \qquad 0 \leq m \leq k-2, \\
E^{-}|k-1\rangle &= \lambda_3|0\rangle \\
K^{\pm 1}|m\rangle &= q^{\pm(\lambda_1-\lambda_2-m)}|m\rangle
\end{aligned} \tag{16}$$

where the complex constants λ_i ($i = 1, 2, 3$) can be determined by the cyclic properties of the k-dimensional representation of the quantum group $U_q(sl(2))$ at a root of unity. Equations (16) imply that the degree of degeneracy of the lowest Landau level is just k, which is the dimension of the considered representation, as in the case of the plane [13] and of the Poincaré half-plane [14].

In summary, we have found that the degeneracy of the lowest Landau level of an electron on the sphere with a magnetic monopole at its center is given by the fundamental cyclic irreducible representation of $U_q(sl(2))$ when the root of unity is imposed. Moreover, we have found that this degree is just k, where k is the magnetic flux through that nonflat surface.

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